

A note on the Gaussian curvature of harmonic surfaces

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Abstract

It was proved that the fundamental group of the space of harmonic polynomials of degree n ($n \geq 2$) which their graphs have the same Gaussian curvature is not trivial. Furthermore, we give an example of topologically nonequivalent conjugate harmonic functions which their graphs have the same Gaussian curvature.

1 Topological equivalency of harmonic functions

Let $w = u(x, y)$ be (at least) a C^2 harmonic function of real variables x and y defined on a region Ω of \mathbb{R}^2 . It is well known that its Gaussian curvature, denoted by $k(x, y)$, can be given by:

$$k(x, y) = \frac{(u_{xx}u_{yy} - u_{xy}^2)}{(1 + u_x^2 + u_y^2)}.$$

Set $z = x + iy$ and rewrite $w = u(x, y)$ as follows:

$$u(x, y) = \operatorname{Re} f(z) = \frac{1}{2}(f(z) + \overline{f(z)}).$$

The partial differential operators of the holomorphic function $f(z)$ of the complex variable z are given by:

$$\frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right),$$

These definitions are forced upon us by the logical requirement that

$$\frac{\partial f}{\partial z} = f'(z), \quad \frac{\partial f}{\partial \bar{z}} = 0, \quad \frac{\partial \bar{f}}{\partial z} = 0, \quad \frac{\partial \bar{f}}{\partial \bar{z}} = \overline{f'(z)}.$$

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It is easily seen that the Gaussian curvature of the graph obtained from $w = u(x, y)$ can be write as:

$$K(x, y) = \frac{|f''(z)|^2}{-(1 + |f'(z)|^2)^2}. \quad (1)$$

We recall that a critical point p of a smooth function $w = g(x_1, x_2)$ is called non-degenerate if the determinant of the Hessian matrix

$$H(g) = \left(\frac{\partial^2 g(x_1, x_2)}{\partial x_i \partial x_j} \right)$$

dose not vanish at p . Otherwise, it is called degenerate critical point.

Lemma 1.1. *The Gaussian curvature of the graph of the harmonic function $w = u(x, y)$, defined on a region Ω is nonpositive, and vanishes only in isolated points which are degenerate critical points of w .*

Proof. Let $f(x + iy)$ be a holomorphic function, and $w = u(x, y)$ its real part. The Gaussian curvature of the graph of $w = u(x, y)$ vanishes when $f''(x + iy) = 0$. Since $f(x, y)$ is holomorphic, it follows that $f''(x + iy)$ is holomorphic therefore, its zeros are isolated. Moreover, critical points of $f(x + iy)$ and $w = u(x, y)$ coincide. The zeros of $f''(x + iy) = 0$ are critical points of $f(x + iy)$ and hence critical points of $w = u(x, y)$. From the equation

$$K(x, y) = \frac{u_{xx}u_{yy} - u_{xy}^2}{(1 + u_x^2 + u_y^2)^2} = 0,$$

it follows that, $u_{xx}u_{yy} - u_{xy}^2 = 0$ if and only if critical points are degenerate. \square

Remark 1.1. *Suppose $w = u(x, y)$ is a harmonic function of degree $n(n > 2)$. Then it is obvious that the cardinality of the set Γ of points at which the Gaussian curvature of the graph of $w = u(x, y)$ are zero, is not greater than $n - 2$. By Gauss-Lucas theorem (cf. [1]) points of Γ lie within the convex hull Δ containing the zeros of holomorphic function $f(x + iy) = u(x, y) + iv(x, y)$. If the multiplicity of zeros of $f(x + iy)$ is not greater than two, then the points of Γ lie inside the convex hull $\delta \subset \Delta$ containing the zeros of $f'(x + iy)$.*

Remark 1.2. *It is immediate from the Equation (1) that if $f = u + iv$ is a holomorphic function, then the graphs of u, v and f have the same Gaussian curvature, see [2].*

Recall that two smooth functions $w, v : \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ are said to be topologically equivalence if there exist homomorphisms $\phi : \Omega \rightarrow \Omega$ and $\psi : \mathbb{R} \rightarrow \mathbb{R}$ such that $\psi \circ w = v \circ \phi$.

Let f be a smooth function on Ω , and p a non-degenerate critical point of f . A component of the level line $f^{-1}(p)$ is called fiber. Two functions f and g are called fiber equivalent at a non-degenerate critical point p if there exists a homeomorphism which maps corresponding fibers to each other, local extrema to local extrema and saddle points to saddle points. Clearly, if f and g are topological equivalent they are fiber equivalent at non-degenerate critical points.

Proposition 1.1. *There exist topological nonequivalent conjugate harmonic functions which their graphs have the same Gaussian curvature.*

Proof. Consider conjugate harmonic functions $u(x, y) = x^3 - 3xy^2 - 3x$ and $v(x, y) = -y^3 + 3x^2y - 3y$. Each of these functions have two non-degenerate critical points $(\pm 1, 0)$. But for the function $u(x, y)$ these critical points lie in different level lines, and for the function $v(x, y)$ critical points lie on one level line. Consequently, they are not topologically equivalent, but their Gaussian curvature is the same because they are conjugate harmonic functions. \square

Remark 1.3. *The graphs of functions $u(x, y)$ and $v(x, y)$ are not isometric, they have different first quadratic forms.*

2 Gaussian Curvature of harmonic polynomials

We denote the Gaussian curvature of the graph a complex polynomial P by G_P :

$$G_P := \frac{|P''(z)|^2}{-(1 + |P'(z)|^2)^2}.$$

Lemma 2.1. *The Gaussian curvature of the graph of harmonic polynomials $u(x, y)$ and $v(x, y)$ of different degrees are always different.*

Proof. Let $Q(z)$ and $P(z)$ be complex polynomials of degree n and m resp., and let $u(x, y)$ and $v(x, y)$ be their real parts, resp. By assumption $m \neq n$. Functions $|P(z)|$ and $|Q(z)|$ attend to $|z^n|$ and $|z^m|$, resp., when z goes to the infinity. Employing this fact for

$$K(x, y) = \frac{|f''(z)|^2}{-(1 + |f'(z)|^2)^2}, \tag{2}$$

we obtain that

$$\frac{|P''(z)|^2}{-(1 + |P'(z)|^2)^2} \neq \frac{|Q''(z)|^2}{-(1 + |Q'(z)|^2)^2}.$$

Thus, the Gaussian curvature of polynomials $u(x, y)$ and $v(x, y)$ are different \square

Theorem 2.1. *Suppose $P(z)$ and $Q(x)$ are complex polynomials. Then G_P and G_Q coincide if and only if $P(z) = \alpha Q(z) + \beta$, where β is a complex constant and α is a complex number such that $|\alpha| = 1$.*

Proof. Sufficiency: If $P(z) = \alpha Q(z) + \beta$, then from the Formula (1) it follows the conclusion of the theorem.

Necessity: We only need to show that

$$\frac{|P''(z)|}{1 + |P'(z)|^2} \quad \text{and} \quad \frac{|Q''(z)|}{1 + |Q'(z)|^2} \quad (3)$$

are equal. Put $P'(z) := n(z)$, obviously $n(z)$ is polynomial. Assume $\gamma(t)$ is a path lies in a region where $n(z)$ is defined, with the initial point z_0 and the end point z . Consider the following integrals

$$\begin{aligned} \int_{\gamma(t)} \frac{|n'(z)||d(z)|}{1 + |n(z)|^2} &= \int_{\gamma(t)} \frac{d(|n(z)|)}{1 + |n(z)|^2} \\ &= \arctan |n(z)| + \text{const.} \end{aligned}$$

Similarly, set $Q'(z) := m(z)$ and perform the same arguments, we obtain

$$\begin{aligned} \int_{\gamma(t)} \frac{|m'(z)||d(z)|}{1 + |m(z)|^2} &= \int_{\gamma(t)} \frac{d(|m(z)|)}{1 + |m(z)|^2} \\ &= \arctan |m(z)| + \text{const.} \end{aligned}$$

Thus $\arctan |n(z)| = \arctan |m(z)| + \beta$, where β is constant, and therefore

$$|n(z)| = |m(z)| + \beta.$$

Now we can easily show that $P(z) = \alpha Q(z) + \beta$, where $|\alpha| = 1$. \square

Remark 2.1. *Theorem 2.1 is not valid for any holomorphic functions, see examples in [2].*

Let $w = P(x, y)$ and $v = Q(x, y)$ be conjugate harmonic polynomials, if the parameter t varies in the unit circle in the complex plane. Then 1-parameter family of polynomials

$$\cos(t)u(x, y) - \sin(t)v(x, y)$$

forms a loop in the space of harmonic polynomials which their graphs have the same Gaussian curvature. It is easily seen that the functions $\pm P(x, y)$ and $\pm Q(x, y)$ lie in the loop. Hence, from Theorem 2.1 and Lemma 1.1 follows the next fact:

Proposition 2.1. *The fundamental group of the space of harmonic polynomials of degree n ($n \geq 2$) which their graphs have the same Gaussian curvature is not trivial.*

References

- [1] M. Marden. The geometry of the zeros of a polynomial in complex variable. American Mathematical Society surveys 3, New York, 1949.
- [2] J. Shomberg. A note on surfaces with radially symmetric nonpositive Gaussian curvature. Mathematica Bohemica. 130, P. 167 - 176, 2005.